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LETTER TO THE EDITOR

On asymptotic continuity of functions of quantum states

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Abstract

A useful kind of continuity of quantum states functions in asymptotic regime is so-called asymptotic continuity. In this letter, we provide general tools for checking if a function possesses this property. First we prove equivalence of asymptotic continuity with so-called *robustness under admixture*. This allows us to show that relative entropy distance from a convex set including a maximally mixed state is asymptotically continuous. Subsequently, we consider *arrowing*—a way of building a new function out of a given one. The procedure originates from constructions of intrinsic information and entanglement of formation. We show that arrowing preserves asymptotic continuity for a class of functions (so-called subextensive ones). The result is illustrated by means of several examples.

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1. Introduction

One of basic issues of quantum information theory is to evaluate operational quantities such as capacities of quantum (usual of teleportation) channel [1, 2] costs creating quantum states under some natural constraints [3, 4], compression rates [5] or localizable information rates [6, 7]. The quantities are usually defined in spirit of Shannon—in an asymptotic regime of many uses of a channel or many copies of a state. Apart from such operational quantities one also considers mathematical functions, that are expected to reflect somehow those features of states or channels. To this end, one chooses functions that satisfy some requirements. For example, most of entanglement measures are mathematical functions that do not increase under local operations and classical communication [3, 8]. Other examples are correlation measures (see, e.g., [9–12]). Such functions turn out to be very useful, as they often provide upper or lower bounds for operational quantities in an asymptotic regime, the functions are especially useful if they are *asymptotically continuous*. The prototype for asymptotic continuity is Fannes inequality [13] for von Neumann entropy $S(\varrho) = -\mathrm{tr}\,\varrho\log\varrho$, which says that for any states ϱ and σ with $\|\varrho-\sigma\|_1 \leqslant 1/2$ we have

$$|S(\varrho) - S(\rho)| \leqslant \|\varrho-\sigma\|_1 \log d + \eta(\|\varrho-\sigma\|_1) \tag{1}$$

where $\eta(x) = -x\log x$, d is the dimension of the Hilbert space. The important feature of this stronger form of continuity is that the right-hand side scales logarithmically with dimension of the Hilbert space. This kind of inequality was first applied to the quantum information theory in [14, 15] to provide a lower bound for compression rates of mixed signal states (interestingly, the question of achievability of the bound is in general still open). Subsequently, it was applied to entanglement theory [16] which leads, in particular, to methods of providing bounds for distillable entanglement and entanglement cost [4, 17]. Asymptotic continuity has become an important tool in proving irreversibility of pure states transformations (see [18] and references therein).

In [19, 20] two measures of entanglement have been proven to satisfy Fannes-like inequality (i.e. to be asymptotically continuous)—entanglement of formation E_F [3] and relative entropy of entanglement [21]. In [22] asymptotic continuity of conditional entropy $S(A|B) = S(\varrho_{AB}) - S(\varrho_B)$ has been proven, where the right-hand side depends only on the dimension of system A. This allowed us to prove asymptotic continuity of third measure of entanglement—squashed entanglement [23]. The importance of asymptotic continuity was made even more transparent in [24] where it was shown that a convex and so-called *subextensive* function, if not asymptotically continuous, behaves in a quite weird way: namely, after removing one qubit, it can change at an arbitrarily large amount.

Clearly, it is very important to know whether a function is asymptotically continuous or not. Yet it is usually rather a difficult task. The aim of this letter is to provide general tools for checking asymptotic continuity. First, we show that the latter is equivalent to so-called 'robustness under admixtures', i.e. a function is asymptotically continuous, if it does not change too much under admixing any state with a small weight. Using it, we prove that relative entropy distance from any convex set including a maximally mixed state is asymptotically continuous, extending therefore the result of [20] where it was proven for compact and convex sets.

Next, we consider a procedure, called *arrowing*, of building new functions out of given functions. The procedure originates both from classical privacy theory [25, 26]—where the prototype was so-called intrinsic information—as well as from entanglement theory, since it includes as a special case the other procedure called *convex roof* [27], the prototype of which was entanglement of formation [3]. Since arrowing is commonly used in different contexts (see quite recent application [28]), it is important to be able to check the properties of arrowed versions of different functions. We provide here a quite general result, showing that for subextensive functions such a procedure preserves asymptotic continuity, i.e. if an original function is asymptotically continuous, so is its 'arrowed' version. We then apply it to show that some tripartite entanglement measure [18, 29] as well as so-called *mixed convex roof* of quantum mutual information introduced in [25] are asymptotically continuous.

2. Basic definitions

In this section, we will introduce some definitions which we will use throughout this letter.

Set of states. A positive operator $\varrho \in \mathcal{S}$ with $\mathrm{tr}\,\varrho = 1$, acting on Hilbert space $\mathcal{H}$ we will call state. A set of all states will be denoted by $\mathcal{S}(\mathcal{H})$. (We will deal with finite-dimensional

Hilbert spaces.) A state is called pure if it is of the form $|\psi\rangle\langle\psi|$ where $\psi \in \mathcal{H}$. Otherwise it is called a mixed state.

Von Neumann entropy $S(\varrho)$ for a state ϱ is given by the following formula:

$$S(\varrho) = -\mathrm{tr}\,\varrho \log \varrho. \qquad (2)$$

We use base 2 logarithm in this letter.

Relative entropy for states ϱ and σ is defined as

$$S(\varrho|\sigma) = \mathrm{tr}\,\varrho \log \varrho - \mathrm{tr}\,\varrho \log \sigma. \qquad (3)$$

Trace norm of an operator A is given by

$$\|A\|_1 = \mathrm{tr}\sqrt{AA^\dagger} \qquad (4)$$

where $A^\dagger$ stands for the Hermitian conjugation.

Measurement. We will consider measurements with a finite number of outcomes, represented by finite sets of operators $\mathcal{M} = \{A_i\}$ satisfying $\sum_i A_i^\dagger A_i = I$. Slightly abusing terminology, we will call the measurements POVMs (positive operator-valued measure).

Subextensivity. A function $f : \mathcal{S}(\mathcal{H}) \to R$ is subextensive if

$$\forall_\varrho \quad \exists_M \quad f(\varrho) \leqslant M \log d \qquad (5)$$

where M is a constant, $d = \dim \mathcal{H}$.

Definition 1. *Let f be a real-valued function $f : S(\mathcal{C}^d) \mapsto \mathcal{R}$ and ϱ_1, ϱ_2 are the states acting on Hilbert space $\mathcal{C}^d$ and $\varepsilon = \|\varrho_1 - \varrho_2\|_1$. Then a function is asymptotically continuous if it fulfils the following condition*

$$\forall_{\varrho_1,\varrho_2} |f(\varrho_1) - f(\varrho_2)| \leqslant K_1 \varepsilon \log d + O(\varepsilon), \qquad (6)$$

where K_1 is a constant and $O(\varepsilon)$ is any function, which satisfies the condition that $O(\varepsilon)$ converges to 0 when ε converges to 0 and depends only on ε. (In particular, it does not depend on dimension.)

Definition 2. *Let f be a real-valued function $f : S(\mathcal{C}^d) \mapsto \mathcal{R}$ and ϱ_1, ϱ_2 are states acting on Hilbert space $\mathcal{C}^d$. Then a function is robust under admixtures if*

$$\forall_{\varrho_1,\varrho_2} \forall_{\delta>0} |f((1-\delta)\varrho_1 + \delta\varrho_2) - f(\varrho_1)| \leqslant K_2 \delta \log d + O(\delta) \qquad (7)$$

where K_2 is a constant and $O(\delta)$ is any function, which satisfies the condition that $O(\delta)$ converges to 0 when δ converges to 0 and depends only on δ. (In particular, it does not depend on dimension.)

Remark. Note that usually for asymptotic continuity or robustness under admixtures we will not require fulfilling conditions (6) and (7) for the whole range of ε or δ. We will rather restrict to some limited subset of a positive real value of ε or δ (limited by 1 or $\frac{1}{2}$, for example).

3. Asymptotic continuity and robustness under small admixtures

In this section, we prove equivalence between asymptotic continuity and robustness under admixtures of function. This is an extension of the result of [24], where it is proved that if a function f, under admixtures, does not change more than a constant, and subextensive then it is also asymptotically continuous.

Proposition 1. *Let f be a function $f : S(\mathcal{C}^d) \mapsto \mathcal{R}$, then the function is asymptotically continuous if only if it is robust under admixtures.*

Remark. This proposition can also be proved when we do not require 'Lipschitz-type' continuities, but rather 'Cauchy-type' ones (see the appendix).

Proof. '⇒' We assume that the function is asymptotically continuous. This implies

$$|f((1-\delta)\varrho_1 + \delta\varrho_2) - f(\varrho_1)| \leqslant K_1\|\varrho_1 - ((1-\delta)\varrho_1 + \delta\varrho_2)\|_1 \log d + O(\|\varrho_1 - ((1-\delta)\varrho_1 + \delta\varrho_2)\|_1) = K_1\|\delta\varrho_1 - \delta\varrho_2\|_1 \log d + O(\|\delta\varrho_1 - \delta\varrho_2\|_1) \leqslant 2K_1\delta \log d + O(2\delta). \quad (8)$$

Let us take $K_2 = 2K_1$. Then

$$|f((1-\delta)\varrho_1 + \delta\varrho_2) - f(\varrho_1)| \leqslant K_2\delta \log d + O(\delta). \quad (9)$$

'⇒'

We will base on the result of [22] (see also [30]), which can be viewed as a sort of generalized Tales theorem,

$$\forall_{\varrho_1,\varrho_2} \quad \exists_{\sigma,\gamma_1\gamma_2} \quad \sigma = (1-\varepsilon)\varrho_1 + \varepsilon\gamma_1 = (1-\varepsilon)\varrho_2 + \varepsilon\gamma_2 \quad (10)$$

where $\varrho_1, \varrho_2, \sigma, \gamma_1, \gamma_2$ are states acting on the Hilbert space and $\varepsilon = \|\varrho_1 - \varrho_2\|_1$. Using it we obtain

$$|f(\varrho_2) - f(\varrho_1)| \leqslant |f(\varrho_2) - f(\sigma)| + |f(\sigma) - f(\varrho_1)| = |f((1-\varepsilon)\varrho_2 + \varepsilon\gamma_2) - f(\varrho_2)| + |f((1-\varepsilon)\varrho_1 + \varepsilon\gamma_1) - f(\varrho_1)| \leqslant 2K_2\varepsilon \log d + 2O(\varepsilon) \quad (11)$$

so that we can take $K_1 = 2K_2$. Then

$$|f(\varrho_2) - f(\varrho_1)| \leqslant K_1\varepsilon \log d + O(\varepsilon). \quad (12)$$

This ends the proof. □

3.1. Application: asymptotic continuity of relative entropy distance from convex set of states

In [20] it was shown that so-called relative entropy distance from a convex, compact set including maximally state $\frac{I}{d}$ is asymptotically continuous. The proof was quite complicated. Here, on the basis of proposition 1, we present a more general result, where we do not require compactness of the set. Moreover our proof is more straight.

Relative entropy of distance $E_R^{\mathcal{D}}$ is defined as follows

$$E_R^{\mathcal{D}}(\varrho) = \inf_{\sigma\in\mathcal{D}} S(\varrho|\sigma) \quad (13)$$

where $\mathcal{D}$ is a convex set of states including a maximally mixed state, $\varrho \in \mathcal{C}^d$.

We start with the following lemma:

Lemma 1. *Relative entropy of distance $E_R^{\mathcal{D}}$ fulfils the following condition*

$$\left|E_R^{\mathcal{D}}((1-\varepsilon)\varrho + \varepsilon\sigma) - E_R^{\mathcal{D}}(\varrho)\right| \leqslant 2\varepsilon \log d + H(\varepsilon) \quad (14)$$

where $H(\varepsilon) = -\varepsilon \log \varepsilon - (1-\varepsilon) \log(1-\varepsilon)$.

Proof. First we show that $E_R^{\mathcal{D}}$ satisfies the following inequality:

$$\sum_k p_k E_R^{\mathcal{D}}(\varrho_k) - E_R^{\mathcal{D}}\left(\sum_k p_k\varrho_k\right) \leqslant S\left(\sum_k p_k\varrho_k\right) - \sum_k p_k S(\varrho_k). \quad (15)$$

This fact was shown for relative entropy distance from separable states in [31], but it is also true for relative entropy distance from any convex set of states. Here we repeat this proof for $E_R^{\mathcal{D}}$ defined in (13). Note that for $\varrho = \sum_k p_k \varrho_k$

$$S(\varrho|\sigma) = S\left(\sum_k p_k\varrho_k|\sigma\right) = \operatorname{tr}\left(\sum_k p_k\varrho_k \log\left(\sum_k p_k\varrho_k\right) - \sum_k p_k\varrho_k \log\sigma\right)$$
$$= \operatorname{tr}\left(\sum_k p_k(\varrho_k\log\varrho_k - \varrho_k\log\sigma + \varrho_k\log\varrho - \varrho_k\log\varrho_k)\right)$$
$$= \sum_k p_k S(\varrho_k|\sigma) + \sum_k p_k S(\varrho_k) - S(\varrho). \tag{16}$$

Let $\sigma \in \mathcal{D}$ be a state such that $E_R^{\mathcal{D}} = S(\varrho|\sigma) - \delta$. Then we can rewrite

$$E_R(\varrho) = \sum_k p_k S(\varrho_k|\sigma) + \sum_k p_k S(\varrho_k) - S(\varrho) - \delta$$
$$\geqslant \sum_k p_k E_R(\varrho_k) + \sum_k p_k S(\varrho_k) - S(\varrho) - \delta. \tag{17}$$

Since by the definition of $E_R^{\mathcal{D}}$ δ can be arbitrarily small, we obtain

$$\sum_k p_k E_R(\varrho_k) - E_R\left(\sum_k p_k\varrho_k\right) \leqslant S\left(\sum_k p_k\varrho_k\right) - \sum_k p_k S(\varrho_k). \tag{18}$$

We use also the fact that [32]

$$S\left(\sum_k p_k\varrho_k\right) \leqslant \sum_k p_k S(\varrho_k) + H(\{p_k\}) \tag{19}$$

and that relative entropy distance is a convex function, which is implied by the convexity of quantum relative entropy in two arguments. Also note that E_R is bounded by $\log d$ because $\mathcal{D}$ includes a maximally mixed state (so $E_R \leqslant S(\varrho|\frac{I}{d}) = \log d - S(\varrho) \leqslant \log d$). Then we have

$$|E_R((1-\varepsilon)\varrho + \varepsilon\sigma) - E_R(\varrho)| = |E_R((1-\varepsilon)\varrho + \varepsilon\sigma) - (1-\varepsilon)E_R(\varrho) - \varepsilon E_R(\sigma)$$
$$- \varepsilon E_R(\varrho) + \varepsilon E_R(\sigma)| \leqslant= |E_R((1-\varepsilon)\varrho + \varepsilon\sigma) - (1-\varepsilon)E_R(\varrho) - \varepsilon E_R(\sigma)|$$
$$+ \varepsilon|E_R(\varrho)| + \varepsilon|E_R(\sigma)| = (1-\varepsilon)E_R(\varrho) + \varepsilon E_R(\sigma) - E_R((1-\varepsilon)\varrho + \varepsilon\sigma)$$
$$+ \varepsilon|E_R(\varrho)| + \varepsilon|E_R(\sigma)| \leqslant S((1-\varepsilon)\varrho + \varepsilon\sigma) - (1-\varepsilon)S(\varrho) - \varepsilon S(\sigma)$$
$$+ \varepsilon\log d + \varepsilon\log d \leqslant H(\varepsilon) + 2\varepsilon\log d \tag{20}$$

This ends the proof. □

Remark. Note that the main feature of $E_R^{\mathcal{D}}$ responsible for robustness under admixtures, is the following:

(1) $E_R^{\mathcal{D}}$ satisfy the following inequality:

$$\left|E_R\left(\sum_k p_k\varrho_k\right) - \sum_k p_k E_R(\varrho_k)\right| \leqslant H(\{p_k\}). \tag{21}$$

(2) $E_R^{\mathcal{D}}$ is bounded by $\log d$.

Lemma 2. *Relative entropy of distance E_R is asymptotically continuous i.e.*

$$|E_R(\varrho) - E_R(\sigma)| \leqslant 4\varepsilon\log d + 2H(\varepsilon) \tag{22}$$

where $H(\varepsilon) = -\varepsilon\log\varepsilon - (1-\varepsilon)\log(1-\varepsilon)$ and $\varepsilon = \|\varrho - \sigma\|_1$.

Proof. E_R is the robust under admixtures so under proposition 1 it is also asymptotically continuous. □

4. Asymptotic continuity of functions built by 'arrowing'

In this section, we consider 'arrowing'—a construction that from given function f creates a new function denoted by $f_\downarrow$. The definition is motivated by intrinsic information and its generalizations [33–35]. The new function $f_\downarrow$ is defined on an enlarged system as follows:

Definition 3. *For any function $f : \mathcal{S}(\mathcal{H}_X) \to R$ acting on states of system X, we define the function $f_\downarrow : \mathcal{S}(\mathcal{H}_X \otimes \mathcal{H}_E) \to R$ as follows*

$$f_\downarrow(\rho_{XE}) = \inf_{\{A_i\}} \sum_i p_i f\left(\rho_X^i\right) \tag{23}$$

where infimum is taken over all finite POVMs $\{A_i\}$ performed on system E and

$$p_i = \mathrm{tr}(I_X \otimes A_i)\rho_{XE}, \qquad \rho_X^i = \frac{1}{p_i} \mathrm{tr}_E\left(I_X \otimes A_i \rho_{XE} I_X \otimes A_i^\dagger\right), \tag{24}$$

i.e. p_i is a probability of outcome i, and ρ_X^i is the state of system X given that outcome i was obtained.

Remark. We can define a modified version of the previous function as follows:

Definition 4. *For any function $f : \mathcal{S}(\mathcal{H}_X) \to R$ acting on states of system X, we define function $f_\uparrow : \mathcal{S}(\mathcal{H}_X \otimes \mathcal{H}_E) \to R$ as follows*

$$f_\uparrow(\rho_{XE}) = \sup_{\{A_i\}} \sum_i p_i f\left(\rho_X^i\right) \tag{25}$$

where supremum is taken over all finite POVMs $\{A_i\}$ performed on system E and

$$p_i = \mathrm{tr}(I_X \otimes A_i)\rho_{XE}, \qquad \rho_X^i = \frac{1}{p_i} \mathrm{tr}_E\left(I_X \otimes A_i \rho_{XE} I_X \otimes A_i^\dagger\right) \tag{26}$$

i.e. p_i is a probability of outcome i, and ρ_X^i is the state of system X given that outcome i was obtained.

All features of $f_\downarrow$ presenting in this letter are also valid for function $f_\uparrow$.

We have the following lemma, which is proven in section 9.

Lemma 3. *The infimum in the definition of $f_\downarrow$ is achievable.*

We will show in this section that the asymptotic continuity and subextensivity of function f implies asymptotic continuity of $f_\downarrow$. Thus in a sense, arrowing preserves asymptotic continuity. Let us stress that all the involved systems are finite dimensional.

We will need the following definition:

Definition 5. *Given a function f defined on states of a system X, we define its conditional version F for a quantum-classical state of a system XE*

$$\rho_{XE}^{\mathrm{qc}} = \sum_i p_i \rho_X^i \otimes |i\rangle_E\langle i| \tag{27}$$

as follows:

$$F\left(\rho_{XE}^{\mathrm{qc}}\right) = \sum_i p_i f\left(\rho_X^i\right). \tag{28}$$

If the quantum-classical state was obtained from state ρ_{XE} by a POVM $\mathcal{M}$ performed on system E we will also use notation $F(\rho_{XE}, \mathcal{M}) \equiv F\left(\rho_{XE}^{\mathrm{qc}}\right)$.

Let us now present the main result of this section.

Proposition 2. *Let f be a function defined on states of system X, which is subextensive and asymptotically continuous. Then function $f_\downarrow$ is also asymptotically continuous. Moreover, the constant under asymptotic continuity condition depends only on the dimension of system X.*

Proof. Let ϱ_{XE} and σ_{XE} be states and $\varepsilon = \|\varrho_{XE} - \sigma_{XE}\|_1$. Let $\mathcal{M}_\varrho = \{A_k^\varrho\}$ and $\mathcal{M}_\sigma = \{A_k^\sigma\}$ be the optimal measurements for ρ and σ respectively (i.e. those achieving infimum in the definition of $f_\downarrow$) where $\sum_k A_k^{\varrho\dagger} A_k^\varrho = I_E$, $\sum_k A_k^{\sigma\dagger} A_k^\sigma = I_E$. For measurement $\mathcal{M}_\sigma$, let p_k and q_k be probabilities of outcomes if a state was ϱ and σ respectively. The resulting states on system X, given that the outcome was k, we will denote by ϱ_k and σ_k respectively. Due to asymptotic continuity (see section 2) we assume that

$$|f(\varrho_{XE}) - f(\sigma_{XE})| \leqslant K\varepsilon \log d_X + O(\varepsilon) \tag{29}$$

and due to subextensivity

$$|f(\rho)| \leqslant M \log d_X \tag{30}$$

for any state ρ on system X, where $d_X = \dim \mathcal{H}_X$ and M and K are the constants. Then we have the following estimate

$$\begin{aligned} f_\downarrow(\varrho_{XE}) - f_\downarrow(\sigma_{XE}) &= F(\varrho_{XE}, \mathcal{M}_\varrho) - F(\sigma_{XE}, \mathcal{M}_\sigma) \leqslant F(\varrho_{XE}, \mathcal{M}_\sigma) - F(\sigma_{XE}, \mathcal{M}_\sigma) \\ &= \sum_k p_k f\left(\varrho_X^k\right) - \sum_k q_k f\left(\sigma_X^k\right) \leqslant \left|\sum_k p_k f\left(\varrho_X^k\right) - \sum_k q_k f\left(\sigma_X^k\right)\right| \\ &= \left|\sum_k p_k f\left(\varrho_X^k\right) - p_k f\left(\sigma_X^k\right) + p_k f\left(\sigma_X^k\right) - q_k f\left(\sigma_X^k\right)\right| \\ &\leqslant \sum_k \left(p_k \left|f\left(\varrho_X^k\right) - f\left(\sigma_X^k\right)\right| + |p_k - q_k| \left|f\left(\sigma_X^k\right)\right|\right) \\ &\leqslant \sum_k p_k \varepsilon_k K \log d_X + \varepsilon M \log d_X + O(\varepsilon) \leqslant K_1 \varepsilon \log d_X + O(\varepsilon) \end{aligned} \tag{31}$$

where $\varepsilon_k = \left\|\varrho_X^k - \sigma_X^k\right\|_1$ and $K_1 = 2K + M$. The last two steps of the above estimate are implied by asymptotic continuity, subextensivity of the function f and the following facts (see [23]):

$$\sum_k |p_k - q_k| \leqslant \varepsilon \tag{32}$$

and

$$\sum_k p_k \varepsilon_k \leqslant 2\varepsilon. \tag{33}$$

The inequality (32) we get via the following estimate

$$\begin{aligned} \sum_k |p_k - q_k| &= \left\|\sum_k p_k |k\rangle\langle k| - \sum_k q_k |k\rangle\langle k|\right\|_1 \\ &\leqslant \left\|\sum_k p_k \varrho_X^k \otimes |k\rangle\langle k| - \sum_k q_k \sigma_X^k \otimes |k\rangle\langle k|\right\|_1 \\ &= \|(I_X \otimes \Lambda_\sigma)\varrho_{XE} - (I_X \otimes \Lambda_\sigma)\sigma_{XE}\|_1 \leqslant \|\varrho_{XE} - \sigma_{XE}\|_1 = \varepsilon, \end{aligned} \tag{34}$$

where Λ_σ is a completely positive map induced by POVM $\mathcal{M}_\sigma$ as follows:

$$\Lambda_\sigma(\cdot) = \sum_k \mathrm{tr}\big[A_k^\sigma(\cdot)A_k^{\sigma\dagger}\big]|k\rangle\langle k|. \tag{35}$$

We have used here the fact that trace norm does not increase under completely positive trace preserving maps [36].

The inequality (33) is proven as follows:

$$\begin{aligned}\varepsilon = \|\varrho_{XE} - \sigma_{XE}\|_1 &\geqslant \sum_k \big\| p_k\varrho_X^k \otimes |k\rangle\langle k| - q_k\sigma_X^k \otimes |k\rangle\langle k|\big\|_1 = \sum_k \big\| p_k\varrho_X^k - q_k\sigma_X^k\big\|_1 \\ &\geqslant \sum_k \big(\big\| p_k\varrho_X^k - p_k\sigma_X^k\big\|_1 - \big\| p_k\sigma_X^k - q_k\sigma_X^k\big\|_1\big) \\ &= \sum_k p_k \big\|\varrho_X^k - \sigma_X^k\big\|_1 - \sum_k |p_k - q_k|) \geqslant \sum_k p_k\varepsilon_k - \varepsilon. \end{aligned} \tag{36}$$

Analogously we can show that

$$\begin{aligned} f_\downarrow(\sigma_{XE}) - f_\downarrow(\varrho_{XE}) &= F(\sigma_{XE}, \mathcal{M}_\sigma) - F(\varrho_{XE}, \mathcal{M}_\varrho) \\ &\leqslant F(\sigma_{XE}, \mathcal{M}_\varrho) - F(\varrho_{XE}, \mathcal{M}_\varrho) \leqslant K_1\varepsilon \log d_X + O(\varepsilon). \end{aligned} \tag{37}$$

Thus we obtain

$$|f_\downarrow(\varrho^{XE}) - f_\downarrow(\sigma^{XE}| \leqslant K_1\varepsilon \log d_X + O(\varepsilon). \tag{38}$$

This ends the proof. □

Remark. In the proof we have used the fact that the infimum in the definition of $f_\downarrow$ is achievable. However it is not essential: the proof that does not use it is very similar to the above one.

Finally, consider modification of the function $f_\downarrow$, where we do not optimize over all POVMs, but only over complete POVMs, for which the operators A_k are of rank one.

Definition 6. *For any function $f : \mathcal{S}(\mathcal{H}_X) \to R$ acting on states of system X, we define function $f_\downarrow^{cpl} : \mathcal{S}(\mathcal{H}_X \otimes \mathcal{H}_E) \to R$ as follows*

$$f_\downarrow^{cpl}(\rho_{XE}) = \inf_{\{A_i\}} \sum_i p_i f\big(\rho_X^i\big) \tag{39}$$

where infimum is taken over all finite POVMs $\{A_i\}$ with elements A_i being of rank one. The notation is the same as in definition 3.

Again, the infimum in the above definition can be achieved; see section 9. We then obtain

Proposition 3. *Let f be a function defined on states of system X, which is subextensive and asymptotically continuous. Then function $f_\downarrow^{cpl}$ is also asymptotically continuous. Moreover, the constant under asymptotic continuity condition depends only on the dimension of system X.*

The proof is analogous to the proof of proposition 2.

5. Applications

5.1. Measure of classical correlation $C_\leftarrow$

This proposition implies asymptotic continuity of measure of classical correlation $C_\leftarrow$ defined as follows [9]:

$$C_\leftarrow(\rho_{AB}) = \max_{B_i^\dagger B_i} S(\rho_A) - \sum_i p_i S\big(\rho_A^i\big) \tag{40}$$

where $B_i^\dagger B_i$ is a POVM performed on subsystem B, $\rho_A^i = \mathrm{tr}_B\big(I \otimes B_i \rho_{AB} I \otimes B_i^\dagger\big)\big/p_i$ is the remaining state of A after obtaining the outcome i on B, and $p_i = \mathrm{tr}_{AB}\big(I \otimes B_i \rho_{AB} I \otimes B_i^\dagger\big)$. Note that we can rewrite $C_\leftarrow$ as

$$C_\leftarrow(\varrho_{AB}) = \max_{B_i^\dagger B_i} \sum_i p_i \left(S\left(\sum_i p_i \varrho_A^i\right) - S\big(\varrho_A^i\big)\right). \tag{41}$$

So $C_\leftarrow$ is a kind of function built by 'arrowing', where $f : \mathcal{S}(\mathcal{H}_A) \to R$ acting on states of system A is of the form

$$f\big(\varrho_A^i\big) = S\left(\sum_i p_i \varrho_A^i\right) - S\big(\varrho_A^i\big). \tag{42}$$

The function f is asymptotically continuous, because entropy von Neumann S possess this feature. So asymptotic continuity of quantity $C_\leftarrow$ follows from proposition 2.

5.2. Intrinsic conditional information

Consider the following function called intrinsic conditional information: $I(X; Y \downarrow E)$ [26] between X and Y given E defined as

$$I(X; Y \downarrow E) = \inf_{P_{\bar{E}|E}} I(X; Y|\bar{E}) = \inf_{P_{\bar{E}|E}} \sum_e p(\bar{e}) I(X; Y|\bar{E} = \bar{e}) \tag{43}$$

where $P_{\bar{E}|E}$ is a classical channel, $I(X; Y|\bar{E} = \bar{e})$ is the mutual information between X and Y given $\bar{E} = \bar{e}$ and $p(\bar{e})$ is the probability that we have outcome $\bar{e}$ on subsystem $\bar{E}$. The quantity $I(X; Y|\bar{E}) = \sum_e p(\bar{e}) I(X; Y|\bar{E} = \bar{e})$ is called conditional information. It is known [37] that infimum in the definition of intrinsic conditional information is achievable. It is enough to take minimum over $P_{\bar{E}|E}$ with the system $\bar{E}$ of size of E.

One easily finds that the intrinsic information is a particular case of 'arrowing'. Indeed, for a given classical channel $P_{\bar{E}|E}$ with conditional probabilities $\{p_{\bar{e}|e}\}$ we consider POVM given by Kraus operator $A_{\bar{e}} = \sum_e \sqrt{p_{\bar{e}|e}} |e\rangle\langle e|$. Now, if we embedded in a natural way our distribution into set of quantum states, then we see that definition 3 reproduces the above quantity.

If we note that the mutual information itself is asymptotically continuous (it is a sum of entropies, each of them being asymptotically continuous due to Fannes inequality (1)), then we will see that the asymptotic continuity of intrinsic conditional information follows from our theorem.

6. Convex roof functions

Here we present asymptotic continuity of functions constructed from other asymptotically continuous function f by means of *convex roof* [27]. We will distinguish between *pure* and *mixed* convex roofs. The pure convex roof is the generalization of the definition of entanglement of formation E_F given in [3]. It was proposed and investigated in [27] and called there just a convex roof.

6.1. Pure convex roof

Definition 7. *For a function f defined on pure states its pure convex roof $\widehat{f}$ is a function defined on all states, given by*

$$\widehat{f}(\varrho) = \inf_{\{p_k,\psi_k\}} \sum_k p_k f(\psi_k) \tag{44}$$

where infimum is taken over all finite pure ensembles $\{p_k, \psi_k\}$, satisfying $\varrho = \sum p_k|\psi_k\rangle\langle\psi_k|$.

It is useful to represent a convex roof in a different way (cf [19]), to make explicit, that operation of a pure convex roof is actually arrowing. Indeed, for any state ϱ acting on Hilbert space $\mathcal{H}_X$ of dimension d_X we can construct its purification i.e. pure state φ_ϱ acting on Hilbert space $\mathcal{H}_X \otimes \mathcal{H}_E$ (with $\dim \mathcal{H}_E = \dim \mathcal{H}_X$) such that

$$\mathrm{tr}_{\mathcal{H}_{anc}} \varphi_\varrho = \varrho. \tag{45}$$

Moreover for any pure decomposition of ϱ, given by $\{p_k, \psi_k\}$ there exists a complete POVM on $\mathcal{H}_{anc}$ which gives such an ensemble on system X, and vice versa: any POVM gives rise to some pure decomposition.

Then we can rewrite $\widehat{f}$ as an infimum over measurements $\mathcal{M}$

$$\widehat{f}(\varrho) = \inf_{\sum p_k|\psi_k\rangle\langle\psi_k|=\varrho} \sum_k p_k f(\psi_k). \tag{46}$$

Consequently, we have

$$\widehat{f}(\varrho_X) = f_{\downarrow}^{cpl}\left(\varphi_{XE}^{\varrho}\right) \tag{47}$$

where the equality holds for arbitrarily fixed purification φ_{XE}^{ϱ} of the state ϱ_X. Having rewritten a pure convex roof in terms of an arrowed function, we can easily prove its asymptotic continuity, by use of proposition 2.

Proposition 4. *Let f be a function, which is subextensive and asymptotically continuous. Then its convex roof $\widehat{f}$ is also asymptotically continuous.*

Proof. We will use following inequalities [38]

$$1 - F(\varrho, \sigma) \leqslant \tfrac{1}{2}\|\varrho - \sigma\|_1 \leqslant \sqrt{1 - F(\varrho, \sigma)} \tag{48}$$

where $F(\varrho, \sigma) = \sqrt{\sqrt{\varrho}\sigma\sqrt{\varrho}}$ is the fidelity [39, 40]. The fidelity can be also expressed as follows

$$F(\varrho, \sigma) = \sup |\langle\psi_\varrho|\psi_\sigma\rangle| \tag{49}$$

where supremum is taken over all ψ_ϱ and ψ_σ which are purifications of states ϱ and σ. The supremum is achievable.

Consider now arbitrary states ϱ and σ let $\varepsilon = \|\varrho - \sigma\|_1$. We want to estimate $\widehat{f}(\varrho) - \widehat{f}(\sigma)$. Since the representation (47) does not depend on the choice of purification, we take such purifications ψ_ϱ and ψ_σ, that

$$F(\varrho, \sigma) = F(\psi_\varrho, \psi_\sigma). \tag{50}$$

Then we have

$$\||\psi_\varrho\rangle\langle\psi_\varrho| - |\psi_\sigma\rangle\langle\psi_\sigma|\|_1 \leqslant 2\sqrt{1 - F(\psi_\varrho, \psi_\sigma)} = 2\sqrt{1 - F(\varrho, \sigma)} \leqslant 2\sqrt{\|\varrho - \sigma\|_1/2} = \sqrt{2\varepsilon}. \tag{51}$$

Since we assume that f is asymptotically continuous and subextensive, we can use proposition 2 to get

$$|\widehat{f}(\varrho) - \widehat{f}(\sigma)| = \left|f_\downarrow^{cpl}(\psi_\varrho) - f_\downarrow^{cpl}(\psi_\sigma)\right| \leqslant K\sqrt{2\varepsilon}\log d_X + O(\sqrt{2\varepsilon}) \tag{52}$$

This ends the proof. □

Remark. Note that however we have here $\sqrt{2\varepsilon}$ instead of ε, but we think that it does not change the essence of the condition referring asymptotic continuity.

7. Mixed convex roof

Analogously to a pure convex roof we can define a *mixed convex roof*.

Definition 8. *Let f be a function and ϱ be a state then we can define a function mixed convex roof $\widehat{f}$ as follows*

$$\widehat{f}(\varrho) = \inf_{\{p_k,\varrho_k\}} \sum_k p_k f(\varrho_k) \tag{53}$$

where infimum is taken over all ensembles $\{p_k, \varrho_k\}$, where $\varrho = \sum p_k\varrho_k$.

Similarly as in the case of a pure convex roof we can show that

$$\widehat{f}(\varrho_X) = f_\downarrow\left(\psi_{XE}^\varrho\right) \tag{54}$$

where, again, ψ_{XE}^ϱ is arbitrarily fixed purification of ϱ_X.

Therefore, with an analogous proof as that of proposition 4, we obtain

Proposition 5. *Let f be subextensive and an asymptotically continuous function then the function mixed convex roof $\widehat{f}$ is also asymptotically continuous.*

8. Applications

8.1. Pure convex roof of measure of entanglement for tripartite pure states

Consider the quantity E [31] which is equal to the sum of measure of entanglement for a bipartite state applied to the subsystem of a tripartite state:

$$E(\varrho_{ABC}) = E_R(\varrho_{AB}) + S(\varrho_C) \tag{55}$$

where S is the von Neumann entropy and $\varrho_{AB} = \mathrm{tr}_C\varrho_{ABC}$, $\varrho_C = \mathrm{tr}_{AB}\varrho_{ABC}$ and E_R is relative entropy distance from a set of separable states. Now, we can consider a pure convex roof of the function E as

$$\widehat{E}(\varrho_{ABC}) = \inf_{\varrho_{ABC}=\Sigma p_k|\psi^k\rangle<\psi^k|_{ABC}} \sum_k p_k E\left(\left|\psi_{ABC}^k\right\rangle\right). \tag{56}$$

Note that E is subextensive and asymptotically continuous because relative entropy distance and entropy possess these features. Thus proposition 4 implies that the convex roof of this function $\widehat{E}$ is also asymptotically continuous.

8.2. Entanglement of formation

Proposition 4 implies asymptotic continuity of entanglement of formation E_F (which was first shown in [41]) defined as [3]

$$E_F(\varrho_{AB}) = \inf_{\varrho_{AB}=\Sigma p_k|\psi_k\rangle\langle\psi_k|} \sum_k p_k S_A(|\psi_k\rangle) \tag{57}$$

where S_A is a von Neumann entropy of subsystem A of state. In the original definition, infimum is taken over all *pure* ensembles, but note that in this case infimum over all ensembles reduces to infimum over pure ensembles. Thus

$$E_F(\varrho_{AB}) = \inf_{\varrho_{AB}=\Sigma p_k\varrho_k} \sum_k p_k S_A(\varrho_k). \tag{58}$$

This is implied by the concavity of von Neumann entropy:

$$\begin{aligned}\sum_k p_k S_A(\varrho_k) &= \sum_k p_A S_A\left(\sum_i q_i^k |\varphi_i^k\rangle\langle\varphi_i^k|\right)\\ &\geqslant \sum_k p_k \sum_i q_i^k S_A(|\varphi_i^k\rangle) = \sum_{k,i} p_k q_i^k S_A(|\varphi_i^k\rangle).\end{aligned} \tag{59}$$

So for every mixed ensemble we can find a pure ensemble which gives no greater value of function E_F than a mixed ensemble.

8.3. Pure and mixed convex roofs of mutual information

Now, we show the example of a function for which pure and mixed convex roofs are not equal to each other. Consider the following functions:

$$\widehat{I_M}(\varrho_{AB}) = \inf_{\varrho_{AB}=\Sigma p_k|\psi_k\rangle\langle\psi_k|} \sum_k p_k I_M(|\psi_k\rangle) \tag{60}$$

$$I_M^{\frown}(\varrho_{AB}) = \inf_{\varrho_{AB}=\Sigma p_k\varrho_k} \sum_k p_k I_M(\varrho_k) \tag{61}$$

where I_M is mutual information $I_M = S_A(\varrho_{AB}) + S_B(\varrho_{AB}) - S(\varrho_{AB})$. In our terminology, the functions are pure and convex roofs of quantum mutual information. The second one was introduced in [25]. Note that for a pure convex roof we have

$$\widehat{I_M}(\varrho_{AB}) = 2 \inf_{\varrho_{AB}=\Sigma p_k|\psi_k\rangle\langle\psi_k|} \sum_k p_k S_A(|\psi_k\rangle) = 2E_F(\varrho_{AB}). \tag{62}$$

Let ϱ_{as} be an antysymmetric state:

$$\varrho_{as} = \frac{1}{d^2-d}(I-V) \tag{63}$$

where V is a unitary flip operator V acting on Hilbert space $\mathcal{C}^d \otimes \mathcal{C}^d$ system defined by $V\phi\otimes\varphi = \varphi\otimes\phi$. We know that [42]

$$E_F(\varrho_{as}) = 1. \tag{64}$$

So $\widehat{I_M}(\varrho_{as}) = 2$. Then we have the following inequality:

$$I_M^{\frown}(\varrho_{as}) \leqslant I_M(\varrho_{as}) = 2\log d - S(\varrho_{as}) = \log\frac{2d}{d-1}. \tag{65}$$

So for $d \geqslant 3$ we have that $I_M^{\frown}(\varrho_{as}) \neq \widehat{I_M}(\varrho_{as})$.

9. Achieving infimum in the definition of arrowing

We prove that in the definition of arrowing the infimum is achievable, so that it can be replaced by minimum. First we prove the following lemma.

Lemma 4. *Let $\{p_i\}$ be a probability distribution; then any convex combination $\sum_i p_i x_i$, where $x_i = (\varrho_i, f(\varrho_i))$, equal to $\sum p_i(\varrho_i, f(\varrho_i))$, can be written as a convex combination $\sum q_i(\varrho_i, f(\varrho_i))$ consisting of $n+1$ (or less) ingredients, where n is a dimension of space on which x_i is acting. So*

$$\sum_i p_i \varrho_i = \sum_{i=1}^{n+1} q_i \varrho_i \qquad \text{and} \qquad \sum_i p_i f(\varrho_i) = \sum_{i=1}^{n+1} q_i f(\varrho_i). \tag{66}$$

Proof. Let $\tilde{f} = \sum_i p_i f(\varrho_i)$ where $\varrho = \sum_i p_i \varrho_i$ is a state acting on Hilbert space $\mathcal{H}$. Let $x_i = (\varrho_i, f(\varrho_i))$ be a point from a convex set $\mathcal{S} = co(\varrho_i, f(\varrho_i))$. Then

$$(\varrho, \tilde{f}) = \left(\sum_i p_i \varrho_i, \sum_i p_i f(\varrho_i)\right) = \sum_i p_i(\varrho_i, f(\varrho_i)) \in \mathcal{S}. \tag{67}$$

Using Caratheodory's theorem, we have that there exists such a set of probability distributions consisting of $n+1$ or less elements that

$$(\varrho, \tilde{f}) = \sum_i q_i(\varrho_i, f(\varrho_i)). \tag{68}$$

So $\varrho = \sum_i q_i \varrho_i$ and $\tilde{f} = \sum_i q_i f(\varrho_i)$. This ends the proof. □

Now, we use above lemma to prove that infimum in the function $f_{\downarrow}(\varrho_{XE})$ is achievable. Let ψ_{AXE} be a purification of state ϱ_{XE}. Then if we make measurement $\mathcal{M}$ on subsystem E we get ensemble $\{p_i, \varrho_i^{AX}\}$ on subsystem AX. Let us define function $\tilde{f}$ such that for any given function f

$$\tilde{f}(\varrho_i^{AX}) = f(\varrho_i^X) \tag{69}$$

where $\varrho_i^X = \mathrm{tr}_A \varrho_i^{AX}$. Then

$$f_{\downarrow}(\varrho^{XE}) = \inf_{\mathcal{M}} \sum_i p_i f(\varrho_i^X) = \inf_{\mathcal{M}} \sum_i p_i \tilde{f}(\varrho_i^{AX}). \tag{70}$$

Note that for the function $\tilde{f}$ and state ψ_{AXE} we can define

$$f_{\downarrow}(\psi_{AXE}) = \inf_{\mathcal{M}} \sum_i p_i \tilde{f}(\varrho_i^{AX}) \tag{71}$$

where we treat subsystem AX as a subsystem and E as a second. Note also that

$$f_{\downarrow}(\psi_{AXE}) = \inf_{\{p_i, \varrho_i^{XE}\}} \sum_i p_i \tilde{f}(\varrho_i^{AE}) \tag{72}$$

because we can always find such a measurement made on subsystem E of state ψ_{AXE}, which give us ensemble $\{q_i, \varrho_i^{AX}\}$.

Then using lemma 4 we know that there exists the other finite ensemble $\{q_i, \varrho_i^{AX}\}$ such that

$$\sum_i p_i \varrho_i^{AX} = \sum_i^{d+1} q_i \varrho_i^{AX} \qquad \text{and} \qquad \sum_i p_i \tilde{f}(\varrho_i^{AX}) = \sum_i^{d+1} q_i \tilde{f}(\varrho_i^{AX}) \tag{73}$$

where d is the dimension of space on which $\sum_i p_i \varrho_i^{AX}$ is acting. So for the function $f_\downarrow(\psi_{AXE})$, infimum over measurement is effectively equal to infimum over a bounded finite set of ensembles, so we have infimum over compact states. This implies that there exists an extremal point belonging to $\mathcal{S}$, so infimum for this function is achievable. If we are looking at formulae (70) and (71) we can see that $f_\downarrow(\psi_{AXE}) = f_\downarrow(\varrho^{XE})$, which implies that for any given state ϱ^{XE} function $f_\downarrow(\varrho^{XE})$ achieves infimum.

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Appendix

Now we will present the other version of proposition 1. We will use Cauchy-type conditions for asymptotic continuity and show that they are also equivalent to robustness under admixtures.

Proposition 6. *Let f be a function; then the following conditions are equivalent:*

$$(1) \quad \forall_{\varepsilon>0} \quad \exists_{\delta>0} \quad \forall_{\varrho,\sigma} \quad \|\varrho-\sigma\|_1 \leqslant \delta \Longrightarrow |f(\varrho)-f(\sigma)| \leqslant K_1\varepsilon \log d + O(\varepsilon) \tag{A.1}$$

$$(2) \quad \forall_{\varepsilon>0} \quad \exists_{\delta>0} \quad \forall_{\varrho,\sigma} \quad |f((1-\delta)\varrho+\delta\sigma)-f(\varrho)| \leqslant K_2\varepsilon \log d + O(\varepsilon), \tag{A.2}$$

K_1, K_2 are constants and $O(x)$ is any function that (i) converges to 0 when x converges to 0 and (ii) depends only on x (so in our particular case, it will not depend on the dimension).

Proof. '$1 \Rightarrow 2$' Let $\varepsilon > 0$ be fixed then there exists such $\delta > 0$ that for any states ϱ and σ, the following conditions is fulfilled:

$$\|\varrho-\sigma\|_1 \leqslant \delta \Longrightarrow |f(\varrho)-f(\sigma)| \leqslant K_1\varepsilon \log d + O(\varepsilon). \tag{A.3}$$

Note that there exists such $\delta_1 = \frac{\delta}{2}$

$$\|\varrho-((1-\delta_1)\varrho+\delta_1\sigma)\|_1 = \delta_1\|\varrho-\sigma\|_1 \leqslant 2\delta_1 = \delta, \tag{A.4}$$

this implies that

$$|f((1-\delta_1)\varrho+\delta_1\sigma)-f(\varrho)| \leqslant K_1\varepsilon \log d + O(\varepsilon) = K_2\varepsilon \log d + O(\varepsilon). \tag{A.5}$$

"$2 \Rightarrow 1$"

Let $\varepsilon > 0$ then there exists such $\delta > 0$ that

$$\forall_{\varrho,\sigma} \quad |f((1-\delta)\varrho+\delta\sigma)-f(\varrho)| \leqslant K_2\varepsilon \log d + O(\varepsilon). \tag{A.6}$$

Let ϱ_1, ϱ_2 be the states that

$$\|\varrho_1-\varrho_2\|_1 = \delta_1 \leqslant \delta. \tag{A.7}$$

Analogously to the proof of theorem 1

$$\exists_{\sigma,\gamma_1\gamma_2} \quad \sigma = (1-\delta_1)\varrho_1+\delta_1\gamma_1 = (1-\delta_1)\varrho_2+\delta_1\gamma_2 \tag{A.8}$$

$$\begin{aligned}|f(\varrho_2)-f(\varrho_1)| &\leqslant |f(\varrho_2)-f(\sigma)|+|f(\sigma)-f(\varrho_1)| = |f(\varrho_2)-f((1-\delta_1)\varrho_2+\delta_1\gamma_2)| \\ &\quad +|f((1-\delta_1)\varrho_1+\delta_1\gamma_1)-f(\varrho_1)| \leqslant 2K_2\log d+2O(\varepsilon) = K_1\log d+O(\varepsilon).\end{aligned} \tag{A.9}$$

This ends the proof. □

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